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An Introduction to The Super-Brownian Motion with Catalytic Medium in Dawson-Fleischmann's Work*

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1 Introduction

In [DF97] D.A. Dawson and K. Fleischmann (1997) have constructed a new type of continuous super-Brownian motion X^{ρ} whose branching does occur only in the presence of the so-called catalysts which evolve themselves as another continuous super-Brownian motion ρ given in advance. In other words, the mathematical situation seems to be supposed that the collision local time $L_{[W,\rho]}$ of an underlying Brownian motion path $W = \{W_s\}$ with the catalytic mass process ρ governs the branching system in the sense of Dynkin's additive functional approach [Dy94]. The notion of collision local time was introduced to investigate the support intersection of two independent Dawson-Watanabe superprocesses. Here the collision local time is constructed in connection with Barlow-Evans-Perkins' work [BEP91]. Moreover, Dawson and Fleischmann have discovered in [DF97] new types of limit behaviors in the one dimension. One is about preservation of the mean for the total mass process, and the other is about persistence of the catalytic process in question. More precisely, one can encounter new phenomena that the total mass process converges to a limit without loss of expectation mass and with a nonzero limiting variance, and also that the catalytic super-Brownian motion, starting with a Lebesgue measure l, converges stochastically to l itself. The aim of this expository article is to introduce these new results established recently by Dawson and Fleischmann, in line with [DF97].

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The notion of catalytic branching comes up in the study of catalytic chemical or bilogical systems from the macroscopic viewpoint. The mathematical model of spatially distributed chemical reactions is given by a catalytic reaction diffusion equation in $\mathbf{R}^d$ with the terminal condition, namely

$$\mathcal{L}u = \frac{\partial u}{\partial s} + \frac{1}{2}\Delta u + \rho_s R(u) = 0, \quad 0 \leq s \leq t, \quad u|_{s=t} = \varphi, \tag{1}$$

where R is the reaction term and ρ_s expresses the spatial density of the catalyst at time s. In this backward formulation (1) the spatial density term is assumed to be given by a continuous measure-valued path : $s \mapsto \rho_s$. Since ρ_s may be quite singular, on a mathematically rigorous basis u should be interpreted as a solution of the corresponding integral equation of the evolution form

$$u(s,t,a) = \int p(t-s, b-a)\varphi(b)db + \int_s^t \int p(r-s, b-a)R(u(r,t,b))dr\rho_r(db) \tag{2}$$

where $p(r,b)$ denotes the transition density of a d-dimensional standard Brownian motion. It is interesting to note that infinities can occur in the reaction term of Eq.(2) if $\rho_t(db) \equiv \zeta(db)$ and the measure ζ charges some polar set. On this account, certain restrictions must be placed on ρ in describing the probabilistic development. There is a remarkable relation between catalytic reaction diffusion equations and both catalytic branching particle systems and superprocesses. This allows to take a probabilistic approach to the investigation of such equations. In addition the viewpoint via branching particle systems can make it possible to interpret the catalytic reaction at the microscopic level.

2 Intuitive Interpretation and Branching Models

2.1 Catalytic Branching

First of all we begin with a system of reactant particles at the microscopic level and consider a spatially density field

$$\rho = \{\rho_t(b); t \geq 0, b \in \mathbf{R}^d\}$$

of a catalyst. Here $\rho_t(b)$ should be regarded as the generalized derivative $\rho_t(db)/db$ at b of the possibly singular measure $\rho_t(db)$. We suppose that, basically, the reactant particles move independently in $\mathbf{R}^d$ according to standard Brownian motions W, except that each particle located at time t at b may die or branch with a critical offspring generating function G at rate proportional to the amount of catalyst $\rho_t(b)$ present at time t at b. It means that newly born particles start at the position of their parent but otherwise move independently.

2.2 Laplace Functional and Log-Laplace Function

Let $N(t)$ denote the random number of reactant particles at time t and $x_i(t)$ the random location of the i-th particle at time t. Then

$$\sum_{i=1}^{N(t)} \delta_{x_i(t)}$$

describes the state of reactant at time t. This system of branching Brownian motions, starting at time s with a single particle at a, is expressed by its transition Laplace functional

$$v(s,t,a) := \mathbf{P}_{s,\delta_a} \exp\left\{-\sum_{i=1}^{N(t)} \varphi(x_i(t))\right\}.$$

It is well known that this $v(s,t,a)$ solves the catalytic reaction diffusion equation of the form

$$-\frac{\partial v(s,t,a)}{\partial s} = \frac{1}{2}\Delta v(s,t,a) + \rho_s(a)\{G(v(s,t,a)) - v(s,t,a)\}, \quad v|_{s=t} = \mathrm{e}^{-\varphi}$$

where φ is a nonnegative measurable function on $\mathbf{R}^d$.

An application of Dynkin's additive functional approach [Dy94] allows to reformulate the above equation as

$$v(s,t,a) = \Pi_{s,a}\left[\exp\{-L(s,t)\}\exp\{-\varphi(W_t)\} + \int_s^t \exp\{-L(s,r)\}G(v(r,t,W_r))L(dr)\right]$$

where $\Pi_{s,a}$ denotes the law of Brownian motion W starting at time s at a, and $L = L_{[W,\rho]}$ is a continuous additive functional of W, called *collision local time* between a Brownian particle with path W and the catalytic medium ρ. This collision local time is heuristically given by

$$L_{[W,\rho]}(s,t) = \int_s^t \int \delta_b(W_r)\rho_r(db)dr.$$

Note that the amount of catalyst $\rho_r(b)$, present at time r at b, met by a reactant particle with path W, may possess a precise meaning by virtue of the term $\delta_r(W_r)\rho_r(db)$, even when ρ_r is a singular measure. This suggests from the microscopic viewpoint that a tagged Brownian particle with path W branches according to a clock given by the additive functional $L_{[W,\rho]}$.

Remark. Formally, this collision local time covers the interesting case where the catalyst consists of diffusion particles, namely, $\rho_t = \sum_i \delta_{\sigma_i(t)}$. Here the i-th catalytic particle has position $\sigma_i(t)$ at time t. Then

$$L_{[W,\rho]}(s,t) = \sum_i \int_s^t \delta_{\sigma_i(r)}(W_r)dr.$$

Notice that this makes sense in dimension $d = 1$ only.

The catalytic super-Brownian motion $X = X^\rho = \{X_t^\rho; t \geq 0\}$ in $\mathbf{R}^d$ is obtained by the so-called high density limit procedure [D93] on the assumption that the offspring distribution

of reactant has a finite variance. Actually the offspring distribution is described by a critical offspring generating function G. For the typical case of a finite variance 2, the catalytic super-Brownian motion X is described by the log-Laplace function

$$v(s,t,a) = -\log \mathbf{P}_{s,\delta_a} \exp\left\{-\int \varphi(b) X_t^\rho(db)\right\}.$$

Remark. The above-mentioned log-Laplace function $v(s,t,a)$ solves a special case of the catalytic reaction diffusion equation in Eq.(1) , namely

$$-\frac{\partial v(s,t,a)}{\partial s} = \frac{1}{2}\Delta v(s,t,a) - \rho_s(a)v^2(s,t,a), \quad s \le t, \quad v|_{s=t} = \varphi.$$

Notice that $\rho_s(a)$ is understood as the generalized density function of the measure $\rho_s(da)$.

2.3 Catalytic Process and Super-Brownian Catalytic Medium

We start with a simple case in which $\rho_t(db) = \gamma db$, with a strictly positive constant γ. In this case X^ρ is the usual critical continuous super-Brownian motion with constant branching rate γ. However, on an applicational basis we encounter various kinds of the catalytic mass ρ : in the case with concentration on a hypersurface the catalytic mass can be possibly a singular measure, and ρ can vary in time in the case of varying medium in which it is described by $\rho_t(db)$, or in the case of random medium it may even be sampled from a stochastic object.

In [DF97] Dawson and Fleischmann have initiated the study of a catalytic branching model X^ρ in which the catalytic mass process ρ evolves itself as a super-Brownian motion with constant branching rate γ. It means that ρ is in fact sampled itself from a continuous super-Brownian motion in $\mathbf{R}^d$ with a constant branching rate $\gamma > 0$. Consequently, ρ serves as a catalytic random medium for $X = X^\rho$. In [DF97] ρ is simply called the *catalytic process* and X^ρ the *catalytic super-Brownian motion* (or *catalytic SBM*). This mathematical model captures indeed the specific particle picture that an X-particle as reactant may branch only if it is in the vicinity of a ρ-particle as catalyst. In other words, branching of a reactant is controlled by the collision local time $L_{[W,\rho]}$ of its Brownian path W with all the Brownian paths of the catalyst. We can say that the model is underlain by the mathematical structure that the occupation density of this X-particle on all the ρ-particles determines the reactant branching.

2.4 The Model with Dimensionwise Distinct Profiles

We consider first the one dimensional case. It is well-known that the continuous SBM ρ lives in the set of absolutely continuous measures. Therefore, there is the Radon-Nikodym densities

$$\rho_t(b) = \frac{\rho_t(db)}{db}, \quad \text{for each} \quad t > 0$$

with respect to the Lebesgue measure db taken at b. And also it exists even as a jointly continuous field $\{\rho_t(b);\ t > 0,\ b \in \mathbf{R}\}$ [KS88]. Thus

$$L_{[W,\rho]}(dr) := \left(\int \delta_b(W_r)\rho_r(db)\right)\rho_r(W_r)dr \tag{3}$$

defines a continuous additive functional $L = L_{[W,\rho]}$ of Brownian motion W. This L is called the Brownian collision local time (BCLT) of ρ.

The situation changes dramatically for dimensions $d \geq 2$, since then the random measures $\rho_t(db)$ are singular [DH79]. Hence we cannot use Eq.(3) for the definition of BCLT of ρ. Note that in dimensions $d \geq 2$ the collision local time of a pair of independent Brownain particles is always zero, because independent Brownian particles do not meet in $d \geq 2$. Nevertheless, in dimensions $d = 2, 3$ Brownian collision local times $L = L_{[W,\rho]}$ exist non-trivially for a super-Brownian catalytic medium ρ. For instance, Evans and Perkins [EV94] treat the case of a finite measure-valued SBM ρ. As a consequence a nondegenerate catalytic SBM X^ρ with a catalyst ρ as a SBM can be constructed in these dimensions, even in the infinite measure case. On this account, the discussion is concentrated on a rigorous construction of the continuous catalytic SBM X^ρ for dimensions $d \leq 3$ in [DF97].

However, for dimensions $d \geq 4$ we encounter the degeneration of BCLT $L = L_{[W,\rho]}$ of ρ to zero, since the closure of the graph of ρ does not intersect with the graph of W [BP94]. According to Dawson and Fleischmann [DF97], "..... the reactants do not *feel* the catalyst, thus cannot branch. Therefore in these higher dimensions, if X^ρ exists it must degenerate to the heat flow."

2.5 Qualitative Behavior

When one asks what the long-term behavior of ρ would be like if it starts with a Lebesgue measure m instead, Dawson(1977) [D77] immediately provides with the right answer: ρ_t suffers local extinction as $t \to \infty$ almost surely if $d = 1$, and stochastically if $d = 2$, whereas in all other dimensions takes place convergence in law to a non-trivial steady state ρ_∞ with expectation m.

In [DF97] Dawson and Fleischmann have also initiated the study of the long-term behavior of the catalytic SBM X^ρ in the one dimensional situation. Surprisingly the results obtained in [DF97] are somewhat different from the well-known results on one dimensional branching models. As a matter of fact, the random measure X_t^ρ converges stochastically as $t \to \infty$ to the starting Lebesgue measure $X_0 = m$ for almost all realizations ρ of the catalytic medium starting with $\rho_0 = m$. This means that here we have "persistence". This phenomenon that there is no loss of intensity in the limit, never happen in one dimensional usual spatial branching models with finite variance, but that occurs only in higher dimensions.

Moreover, They also prove in [DF97] persistence of the total mass process in one dimensional case. In the finite measure-valued SBM with constant branching rate, the total mass process is nothing but the Feller critical branching diffusion. In this case the total mass process dies almost surely in a finite time. On the other hand, according to [FLG95] the total mass process converges to 0 with probability one as time tends to infinity in the single point catalytic SBM, where the model is meaningful only in dimension $d = 1$. In contrast to both cases, for the catalytic SBM X^ρ in question, almost sure convergence of the total

mass is proved with preservation of the mean (persistence) and with a nondegenerate limit (nonzero variance).

3 Notations

Let E be a topological space, and $\mathcal{B}(E)$ the Borel-σ-field on E. $f \in \mathcal{B}(E)$ means that f is a real-valued measurable function on E. We write $b\mathcal{B}(E)$ for the subspace of all bounded functions on E. In the case of $E = \mathbf{R}^d$ $(d \geq 1)$, we simply write $\mathcal{B}$, $b\mathcal{B}$. In most cases we work with the space $E = I \times \mathbf{R}^d$ with a finite closed interval $I = [L, T]$ of $\mathbf{R}_+$, $L \leq T$.

For a positive constant $p(> d)$ the reference function φ_p is defined by

$$\varphi_p := (1 + |a|^2)^{-p/2}, \quad a \in \mathbf{R}^d.$$

$\mathcal{B}^p$ denotes the set of all functions $f \in \mathcal{B}$ satisfying

$$|f| \leq C(f)\varphi_p \quad \text{for some constant } C(f).$$

Thses are called functions with p-potential decay. Furthermore, let $\mathcal{B}^{p,I}$ denote the set of all functions $g \in \mathcal{B}(I \times \mathbf{R}^d)$ such that

$$|g(s,\cdot)| \leq C(g)\varphi_p, \quad s \in I, \quad \text{for some constant } C(g).$$

Write $\mathcal{C}$ for the subset of all continuous functions in $\mathcal{B}$, and $\mathcal{C}^p$ (resp. $\mathcal{C}^{p,I}$) is the counterpart space for $\mathcal{B}^p$ (resp. $\mathcal{B}^{p,I}$). Respectively, $\mathcal{B}^p$ (or $\mathcal{B}^{p,I}$) is equipped with the norm

$$\|f\| := \|f/\varphi_p\|_\infty, \quad (\text{or } \|g\|_I := \sup_{s \in I} \|g(s,\cdot)\| \ ), \quad f \in \mathcal{B}^p \ (\text{or } g \in \mathcal{B}^{p,I} \ ).$$

As to the norm for a family of $\mathcal{C}$-spaces, the same story. Those are all Banach spaces.

We introduce the dual set $\mathcal{M}_p$ of all locally finite nonnegative measures μ on $\mathbf{R}^d$ such that

$$\|\mu\|_p := \langle \mu, \varphi_p \rangle < +\infty, \quad \text{with} \quad \langle \mu, \varphi \rangle := \int \varphi(b)\mu(db).$$

The element of $\mathcal{M}_p$ is called a p-tempered measure, and we endow this set $\mathcal{M}_p$ with the p-vague topology. On the other hand, the set $\mathcal{M}_F$ of all finite measures on $\mathbf{R}^d$ is endowed with the weak topology. We denote by $\|\mu\|$ the total mass of μ. In what follows we always denote by ψ_p the function on $I \times \mathbf{R}^d$ which equals φ_p constantly in time. In analogy to $\mathcal{M}_p$, we introduce the set $\mathcal{M}_p^I$ of all measures ν on $I \times \mathbf{R}^d$ satisfying

$$\langle \nu, \psi_p \rangle_I := \int \int_{I \times R^d} \psi_p(r,b) d\nu(r,b) < +\infty.$$

The set $\mathcal{M}_p^I$ is furnished with the weakest topology such that the maps : $\nu \mapsto \langle \nu, \psi \rangle_I$ are continuous for all $\psi \in \mathcal{C}_*^{p,I}$, where $\mathcal{C}_*^{p,I}$ is the subspace of $\mathcal{C}^{p,I}$ of those functions g such that the maps : $a \mapsto g(a)\varphi_p(a)$ can be extended to a function in $\mathcal{C}(I \times \bar{\mathbf{R}}^d)$. The open ball in $\mathbf{R}^d$ with center a, radius r is denoted by $B(a,r)$.

4 Branching Rate Functional

Let $W = \{W_s, \Pi_{s,a}, s \geq 0, a \in \mathbf{R}^d\}$ be a standard Brownian motion in $\mathbf{R}^d$, on canonical path spaces of continuous functions. $p = p(t,a,b)$ denotes its continuous transition density function for $t > 0$, $a, b \in \mathbf{R}^d$ and we write $S = \{S_t; t \geq 0\}$ for the related Brownian semigroup.

$$\Pi_{s,\mu} := \int \Pi_{s,a}\mu(da), \quad s \geq 0, \quad \mu \in \mathcal{M}_p$$

is the law of W starting at time s in the point a distributed according to the infinite measure μ.

Definition 1 *A nonnegative functional $A = A_{[W]}$ of W is called "additive" if, given W, (i) it is a measure $A(dr)$ on $\mathbf{R}^+ := (0,\infty)$; (ii) it is finite on bounded subintervals; (iii) $AJ \equiv A(J)$ is measurable with respect to the universal completion of the σ-field generated by $\{W_r; r \in J\}$, for every $J := (s,t)$, $0 \leq s < t$.*

Definition 2 *An additive functional $K = K_{[W]}$ of W is called a "branching rate functional" if*
(a) it is continuous, i.e., $K(dr)$ carries no mass at any single point set;
(b) it is locally admissible, i.e.,

$$\sup_{a\in\mathbf{R}^d} \Pi_{s,a}\int_s^t \varphi_p(W_r)K(dr) \to 0, \quad (as \ \ s,t \to r_0), \quad r_0 > 0.$$

We denote by $\mathbf{K}$ the set of all branching rate functionals, and $\mathbf{K}_0$ is the subset of $\mathbf{K}$ satisfying (b) with φ_p replaced by the constant function 1.

Remark. Note that $\mathbf{K}_0$ is dense in $\mathbf{K}$. Moreover, $K(dr)$ belongs to $\mathbf{K}$ if and only if $\varphi_p(W_r)K(dr)$ belongs to $\mathbf{K}_0$.

It is easy to show that each $K \in \mathbf{K}$ has uniformly locally finite characteristic:

$$\sup_{(s,a)\in[0,t)\times\mathbf{R}^d} \Pi_{s,a}\int_s^t \varphi_p(W_r)K(dr) < \infty, \quad t > 0. \tag{4}$$

Definition 3 *We say that $K \in \mathbf{K}$ belongs to $\mathbf{K}^*$ if for each $I = [L,T] \subset \mathbf{R}^+$ there exists a constant C_I such that*

$$\sup_{s\in I} \Pi_{s,a}\int_s^T \varphi_p^2(W_r)K(dr) \geq C_I\varphi_p(a), \quad a \in \mathbf{R}^d.$$

Definition 4 *We say that K belongs to $\mathbf{K}^\xi$, ($\xi > 0$) if for each $N > 0$ there is a constant $C_N > 0$ such that*

$$\Pi_{s,a}\int_s^t \varphi_p^2(W_r)K(dr) \leq C_N|t-s|^\xi\varphi_p(a), \quad 0 \leq s \leq t \leq N, \quad a \in \mathbf{R}^d.$$

The followings are two typical examples of K. The first one is a special K in the constant branching rate case. The second is given in connection with the single point catalyst model with dimension $d = 1$.

Example 1 *The special functional $K(dr) \equiv \gamma dr$ is nonrandom and homogeneous in both time and space. This functional is contained in $\mathbf{K}_0 \cap \mathbf{K}^\xi$ with $\xi = 1$.*

Example 2 *In the one dimensional single point catalyst model, the branching rate functional $K(dr)$ is given by the Brownian local time $L^c(dr) = \delta_c(W_r)dr$ at a fixed $c \in \mathbf{R}$. This belongs to $\mathbf{K}_0 \cap \mathbf{K}^\xi$ with $\xi = 1/2$ [DF94].*

5 SBM with Branching Rate Functional K

Let us introduce an $\mathcal{M}_p$-valued critical super-Brownian motion $X = X^K$ with branching rate functional $K \in \mathbf{K}$, which is an important underlying basic process for construction of catalytic SBM in [DF97]. The next proposition guarantees the existence of infinite measure-valued SBM X^K.

Proposition 1 *Let $K = K_{[W]} \in \mathbf{K}$. There exists a time-inhomogeneous $\mathcal{M}_p$-valued Markov process $X = X^K = \{X,\ P_{s,\mu}, s > 0,\ \mu \in \mathcal{M}_p\}$ with Laplace transition functional*

$$P_{s,\mu} \exp\langle X_t, -\varphi\rangle = \exp\langle \mu, -v(s,t,\cdot)\rangle \quad 0 \le s \le t,\ \mu \in \mathcal{M}_p,\ \varphi \in \mathcal{B}^p_+, \tag{5}$$

where $v(\cdot,t,\cdot) \ge 0$ is uniquely determined as solution of the log-Laplace equation

$$v(s,t,a) = \Pi_{s,a}\left[\varphi(W_t) - \int_s^t v^2(r,t,W_r)K(dr)\right], \quad 0 \le s \le t, \quad a \in \mathbf{R}^d. \tag{6}$$

For $0 \le s \le t_1, t_2$, $\mu \in \mathcal{M}_p$, and $\varphi, \psi \in \mathcal{B}^p_+$, we have the covariance formula

$$\mathrm{Cov}_{s,\mu}\left[\langle X_{t_1}, \varphi\rangle, \langle X_{t_2}, \psi\rangle\right] = 2\Pi_{s,\mu}\int_s^{t_1\wedge t_2} S_{t_1-r}\varphi(W_r) S_{t_2-r}\psi(W_r)K(dr).$$

Notice that the covariance could be infinite. Dynkin constructed in [Dy94] an $\mathcal{M}_F$-valued Markov process under restricted conditions on K.

6 The Catalytic Process ρ

The choice of branching rate functional $K(dr) \equiv \gamma dr$ is the well understood special case, in which each corresponding X-particle branches with the constant rate $\gamma > 0$. Then, for $\varphi \in \mathcal{C}^p_+$ and $t > 0$, the solution $v = v(\cdot,t,\cdot)$ of the log-Laplace equation in Eq.(6) of Proposition 1 uniquely solves the paraolic equation

$$-\frac{\partial v}{\partial s} = \frac{1}{2}\Delta v - \gamma v^2, \quad v|_{s=t} = \varphi.$$

The corresponding $\mathcal{M}_p$-valued Markov process $X = X^{\gamma dr}$ is time-homogeneous, which was first constructed by Iscoe(1986) [Is86]. Let us denote by X^K a SBM with branching rate functional $K \in \mathbf{K}^\xi$. For $t \geq s$,

$$Z_t := P_{s,\mu} X_t^K - X_t^K$$

denotes its centered process. Let $\mathcal{D}_0$ denote a countable subset of the domain of the generator $\Delta/2$ of the strongly continuous Brownian semigroup S acting on C_*^p. We define a metric d_p on $\mathcal{M}_p$ by

$$d_p(\mu,\nu) := \sum_{m=1}^{\infty} \frac{1}{2^m} \left(1 \wedge |\langle \mu, \varphi_m \rangle - \langle \nu, \varphi_m \rangle|\right), \quad \mu, \nu \in \mathcal{M}_p.$$

The next theorem is one of the main results in [DF97](cf. Theorem 1, p.234, §3.4).

Theorem 1 (Hölder Continuity of SBM) *Let $K \in \mathbf{K}^\xi$ for some $\xi > 0$. For $N > 0$, $\mu \in \mathcal{M}_p$, $k \geq 1$, and $\varepsilon \in (0, \xi/2)$, there is a modification $\tilde{Z}$ of the centered process Z such that*

$$\sup_{0 \leq s \leq N} \mathbf{P}_{s,\mu} \left[\sup_{s \leq t \leq t+h \leq N} |\langle \tilde{Z}_{t+h} - \tilde{Z}_t, \varphi \rangle| h^{-\varepsilon} \right]^k < +\infty, \quad \varphi \in \mathcal{D}_0. \tag{7}$$

In particular, $P_{s,\mu}$-almost surely, $\tilde{Z}$ has locally Hölder continuous paths of order ε in the metric d_p.

We have $P_{s,\mu} X_t^K = S_{t-s}\mu$ and the map : $t \mapsto S_t\mu \in \mathcal{M}_p(\mathbf{R}^d)$ is continuous. Since $K \in \mathbf{K}^\xi$, we have only to set $\tilde{X}_t = S_{t-s}\mu - \tilde{Z}_t$, $t \geq s$ to get a continuous $\mathcal{M}_p(\mathbf{R}^d)$-valued process. Consequently there is a modification $\tilde{X}$ of the super-Brownian motion $X = X^K$ of Proposition 1 in Section 5 with continuous paths. On this account, it follows that $X^{\gamma dr}$ is continuous (cf. [KS88]).

In [DF97] this particular continuous super-Brownian motion $X^{\gamma dr}$ is used to govern the branching in the catalytic SBM. For convenience, we write simply ρ instead of $X^{\gamma dr}$, and $\mathbf{P}_\mu$ instead of $P_{0,\mu}$ in this case $K(dr) = \gamma dr$. Then we call ρ the catalyst process. In addition, the existence of a jointly Hölder continuous occupation density field related to the catalyst process ρ, in dimensions $d \leq 3$, is established under the supposition that the initial state ρ_0 is not too irregular (cf. Theorem 2, p.252 and Theorem 3, p.254 in §4.6, [DF97]).

7 Brownian Collision Local Time and Catalytic SBM

For $N > 0$, $\varepsilon \in (0, 1]$, and $\eta \in C(\mathbf{R}_+; \mathcal{M}_p)$, set

$$h(\eta, \varepsilon, N) := \sup_{0 \leq s \leq N, a \in R^d} \int_s^{s+\varepsilon} \langle \eta_r, \varphi_p p(r-s, a, \cdot) \rangle dr. \tag{8}$$

Intuitively, the integral in Eq.(8) measures the ε-accumulated density of $\varphi_p \eta$ at (s, a).

Definition 5 (Regular $\mathcal{M}_p$-valued paths) *A path η in $C(\mathbf{R}+;\mathcal{M}_p)$ is called "regular" if $h(\eta,\varepsilon,N)\to,\ 0$ (as $\varepsilon\downarrow 0$) for all $N>0$.*

Roughly speaking, η is regular as far as the ε-accumulated densities of the finite measure-valued path $\varphi_p\eta$ disappear as $\varepsilon\downarrow 0$ uniformly on $[0,N]\times\mathbf{R}^d$ for each $N>0$.

Definition 6 *For a fixed regular path η and $\varepsilon\in(0,1]$, define a continuous additive functional $L^\varepsilon=L^\varepsilon_{[W,\eta]}$ of the Brownian motion W by*

$$L^\varepsilon_{[W,\eta]}:=\langle\eta_r,p(\varepsilon,W_r,\cdot)\rangle dr. \tag{9}$$

We interpret L^ε as the collision local time of η with the ε-vicinity of the Brownian path W. Then we have the following proposition on the existence of Brownian collision local time.

Proposition 2 *For a regular $\mathcal{M}_p$-valued path, there exists an additive functional $L=L_{[W,\eta]}$ of the Brownian motion W such that*
(a) for a strictly positive function ψ in $\mathcal{C}^{p,[0,N]}$ and $N>0$,

$$\sup_{\substack{0\le s\le N\\ a\in R^d}}\Pi_{a,s}\sup_{s\le t\le N}\left|\int_s^t\psi(r,W_r)L^\varepsilon(dr)-\int_s^t\psi(r,W_r)L(dr)\right|^2\to 0,\quad as\quad \varepsilon\downarrow 0;$$

(b) L belongs to $\mathbf{K}$.

The above-mentioned additive functional $L_{[W,\eta]}$ is called the Brownian collision local time (BCLT) of η if η is a regular $\mathcal{M}_p$-valued path. In dimension $d=1$, for all continuous $\mathcal{M}_p$-valued paths η, the BCLT $L=L_{[W,\eta]}$ of η is a branching functional in $\mathbf{K}^\xi$ with $\xi=1/2$ (cf. Corollary 2, p.257 in §5.2, [DF97]).

Next we refer to the Brownian collision local time of the catalyst process. Before stating the result, we need some additional notations. Define

$$q(s,t,a,b):=\int_s^t p(r,a,b)dr,\quad 0\le s\le t,\quad a,b\in\mathbf{R}^d.$$

q is the inhomogeneous Brownian potential kernel, associated with the occupation time $Y^K_{[s,t]}$. Actually, $Y_{[s,t]}$ is a measure on $\mathbf{R}^d$, defined by the process X^K distributed according to $P_{s,\mu}$, $\mu\in\mathcal{M}_p$. Write

$$\mu*q(s,t,b):=\int q(s,t,a,b)\mu(da),\quad \mu\in\mathcal{M}_p,\ 0\le s\le t,\ b\in\mathbf{R}^d.$$

Theorem 2 (Theorem 4, p.259 in §5.3, [DF97]) *Let $d\le 3,\xi\in(0,1/4)$, and $\delta\ge 0$. If $\delta=0$, assume additionally that the map : $[r,z]\mapsto\rho_0*q(0,r,z)$ is locally ξ-Hölder continuous on $\mathbf{R}^+\times\mathbf{R}^d$, with $\mathbf{P}$-probability one, with Hölder constants proportional to $\|\mu\|_p=\langle\mu,\varphi_p\rangle$. Then $\mathbf{P}$-almost surely, the Brownian collision local time $L=L_{[W,\rho_{\delta+(\cdot)}]}$ exists and is a branching rate functional in $\mathbf{K}^\xi$.*

Remark. Recall that $\mathbf{P}$ refers to the catalyst process starting with a spatial homogeneous initial state ρ_0 such that $\|\rho_0\|_p$ has finite moments of all orders. Note that the case $\delta > 0$ covers the time-stationary $\mathbf{P}$ in dimension 3 corresponding to ergodic steady states.

We assume that $d \leq 3$ in what follows, and consider the catalyst process ρ distributed according to $\mathbf{P}$ which is assumed to be either $\mathbf{P}_m$ with a Lebesgue measure m or an ergodic time-stationary law in dimension $d = 3$. Note that in the latter case $\rho_{\delta+(\cdot)}$ is again distributed according to $\mathbf{P}$, for each $\delta > 0$. Hence, an application of Theorem 2 allows to obtain, in both cases, the $\mathbf{P}$-a.s. existence of the BCLT $L = L_{[W,\rho]}$ that is contained in $\mathbf{K}^\xi$ as a branching rate functional, for all $\xi < 1/4$.

Definition 7 (Catalytic SBM) *If the branching rate functional K is $\mathbf{P}$-a.s. given by the BCLT $L = L_{[W,\rho]}$ of ρ, then we write X^ρ for the continuous SBM X^K according to Theorem 1 in Section 6, and $P^\rho_{s,\mu}$, $s \geq 0$, $\mu \in \mathcal{M}_p$, for the quenched distributions of X^ρ given ρ. We call X^ρ the catalytic SBM in the catalytic medium ρ distributed by $\mathbf{P}$.*

8 The Long-Term Behavior of the Catalytic SBM

The following theorem is one of the principal results which asserts the persistence of the total mass process in dimension one (cf. Theorem 5, p.268 in §6.4, [DF97]).

Theorem 3 (Total mass persistence) *Let $d = 1$. For $\mathbf{P}_m$-almost all realizations ρ of the catalyst process, and for $\mu \in \mathcal{M}_p$ and $s \geq 0$,*
(a) $m^\rho := \lim_{t\to\infty} \|X^\rho_t\|$ exists $P^\rho_{s,\mu}$-a.s. ;
(b) The limiting total mass m^ρ has the Laplace function

$$P^\rho_{s,\mu} \exp\{-\theta m^\rho\} = \exp\langle \mu, -u_\theta(s)\rangle, \quad \theta \geq 0$$

with $u_\theta \geq 0$ such that the Feynman-Kac identity

$$u_\theta(s,a) = \theta \Pi_{s,a} \exp\left\{-\int_s^\infty \rho_r(W_r) u_\theta(r, W_r) dr\right\}$$

holds for $s \geq 0$, $a \in \mathbf{R}$.

The last main result in [DF97] is about the persistence in the infinite measure case in dimension one. Starting X^ρ with a Lebesgue measure, opposed to other one-dimensional spatial branching processes, the catalytic SBM X^ρ does not become locally extinct and is even persistent.

Theorem 4 (cf. Theorem 6, p.273, in §6.5, [DF97]) *In dimension $d = 1$, for $\mathbf{P}_m$-almost all realization ρ of the catalyst process, the catalytic SBM X^ρ converges to the Lebesgue measyre m, $P^\rho_{s,m}$-stochastically, as t tends to infinity, for $s \geq 0$.*

References

[BEP91] Barlow, M.T., Evans, S.N. and Perkins, E.A. : Collision local times and measure-valued processes, *Can. J. Math.* **43**(1991), 897-938.

[BP94] Barlow, M.T. and Perkins, E.A. : On the filtration of historical Brownian motion, *Ann. Prob.* **22**(1994), 1273-1294.

[D77] Dawson, D.A. : The critical measure diffusion process, *Z. W. verw. Gebiete* **40**(1977), 125-145.

[D93] Dawson, D.A. : Measure-valued Markov processes, *LNM* **1541**(1993, Springer-Verlag), 1-260.

[DF94] Dawson, D.A. and Fleischmann, K. : A super-Brownian motion with a single point catalyst, *Stoch. Proc. Appl.* **49**(1994), 3-40.

[DF97] Dawson, D.A. and Fleischmann, K. : A continuous super-Brownian motion in a super-Brownian medium, *J. Theor. Prob.* **10**(1997), 213-276.

[DH79] Dawson, D.A. and Hochberg, K.J. : The carrying dimension of a stochastic measure diffusion, *Ann. Prob.* **7**(1979), 693-703.

[Dy94] Dynkin, E.B. : *An Introduction to Branching Measure-Valued Processes*, AMS, Providence, 1994.

[EP94] Evans, S.N. and Perkins, E.A. : Measure-valued branching diffusions with singular interactions, *Can. J. Math.* **46**(1994), 120-168.

[FLG95] Fleischmann, K. and Le Gall, J.-F. : A new approach to the single point catalytic super-Brownian motion, *Prob. Th. Rel. Fields* **102**(1995), 63-82.

[Is86] Iscoe, I. : A weighted occupation time for a class of measure-valued critical branching Brownian motions, *Prob. Th. Rel. Fields* **71**(1986), 85-116.

[KS88] Konno, N. and Shiga, T. : Stochastic partial differential equations for some measure-valued diffusions, *Prob. Th. Rel. Fields* **79**(1988), 201-225.